

# Unobstructed deformations of generalized complex structures induced by $C^\infty$ logarithmic symplectic structures and logarithmic Poisson structures

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## Abstract

We shall introduce the notion of  $C^\infty$  logarithmic symplectic structures on a differentiable manifold which is an analog of the one of logarithmic symplectic structures in the holomorphic category. We show that the generalized complex structure induced by a  $C^\infty$  logarithmic symplectic structure has unobstructed deformations which are parametrized by an open set of the second de Rham cohomology group of the complement of type changing loci if the type changing loci are smooth. Complex surfaces with smooth effective anti-canonical divisors admit unobstructed deformations of generalized complex structures such as del pezzo surfaces and Hirzebruch surfaces. We also give some calculations of Poisson cohomology groups on these surfaces. Generalized complex structures  $\mathcal{J}_m$  on the connected sum  $(2k-1)\mathbb{C}P^2 \# (10k-1)\overline{\mathbb{C}P^2}$  as in [CG1], [GH] are induced by  $C^\infty$  logarithmic symplectic structures modulo the action of  $b$ -fields and it turns out that generalized complex structures  $\mathcal{J}_m$  have unobstructed deformations of dimension  $12k + 2m - 3$ .

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# 1 Introduction

Generalized complex structures are mixed geometric structures building a bridge between complex geometry and real symplectic geometry. Both complex structures and real symplectic structures give rise to generalized complex structures of special classes. However a generalized complex structure on a manifold can admit the type changing loci on which the type of the structure can change from the one from a real symplectic structure to the one from a complex structure.

A complex surface  $S$  with a non trivial holomorphic Poisson structure  $\beta$  has a generalized complex structure  $\mathcal{J}_\beta$  with type changing loci at zeros of  $\beta$ . It is striking that  $(3k-1)\mathbb{C}P^3 \# (10k-1)\overline{\mathbb{C}P^2}$  does not admit complex structures and real symplectic structures, but  $(3k-1)\mathbb{C}P^3 \# (10k-1)\overline{\mathbb{C}P^2}$  has generalized complex structures  $\mathcal{J}_m$  with  $m$  type changing loci [CG1], [GH].

The deformation complex of a generalized complex structure is given by the Lie algebroid complex  $(\wedge^\bullet \overline{L}_{\mathcal{J}}, d_L)$ . Then the space of infinitesimal deformations is the second cohomology group  $H^2(\wedge^\bullet \overline{L}_{\mathcal{J}})$  and the obstruction space is the third cohomology group  $H^3(\wedge^\bullet \overline{L}_{\mathcal{J}})$ . Then the Kuranishi families of generalized complex structures are constructed [Gu1].

A  $C^\infty$  logarithmic symplectic structure  $\omega_C$  on a manifold  $M$  along a submanifold  $D$  of real codimension 2 is a complex 2-form which is given by the following on a neighborhood of  $D$ ,

$$\omega_C = \frac{dz_1}{z_1} \wedge dz_2 + dz_3 \wedge dz_4 + \cdots + dz_{2m-1} \wedge dz_{2m},$$

where  $(z_1, \dots, z_{2m})$  are complex coordinates and  $D = \{z_1 = 0\}$ . On a neighborhood of the complement  $M \setminus D$ ,  $\omega_C$  is a smooth 2-form  $b + \sqrt{-1}\omega$  where  $b$  is a  $d$ -closed 2-form and  $\omega$  denotes a real symplectic structure. (Note that we do not assume that  $M$  is a complex manifold.) The exponential  $\phi := e^{\omega_C}$  gives rise to a generalized complex structure  $\mathcal{J}_\phi$

One of the purposes of the paper is to show that the generalized complex structure  $\mathcal{J}_\phi$  induced by a  $C^\infty$  logarithmic symplectic structure  $\omega_C$  has unobstructed deformations which are given by the second cohomology group of the complement of type changing loci if type changing loci are smooth (see Theorem 6.2 and Theorem 6.3). Our unobstructedness theorems may be regarded as an analog of the unobstructedness theorems of Calabi-Yau and hyperKähler manifolds. However we do not use the Hodge theory and the  $\partial\bar{\partial}$ -lemma but our method is rather topological.

The generalized complex structure  $\mathcal{J}_\beta$  on a Poisson surface  $S$  and  $\mathcal{J}_m$  on the  $(3k-1)\mathbb{C}P^3 \# (10k-1)\overline{\mathbb{C}P^2}$  are induced by  $C^\infty$  logarithmic symplectic structures modulo the action of  $b$ -fields. Thus we can apply our unobstructedness theorems to these generalized complex structures (see Theorem 9.2 and Theorem 9.19).

The paper is organized as follows. In Section 2 we give a short explanation of generalized complex structures and in Section 3 we also provide fundamental notions of deformations of generalized complex structures. It is remarkable that Poisson cohomology groups of a Poisson structure  $\beta$  which are the hypercohomology groups of the Poisson complex coincide with the Lie algebroid cohomology groups of the generalized complex structure  $\mathcal{J}_\beta$  induced by  $\beta$ . In Section 4, we discuss the generalized complex structures induced from a logarithmic symplectic structure along a smooth divisor  $D$  on a complex manifold  $X$  which is the dual of a holomorphic Poisson structure  $\beta$ . In Section 5, we show that the Poisson cohomology groups of  $\beta$  are isomorphic to the de Rham cohomology

groups of the complement  $X \setminus D$ . In Section 6, we introduce  $C^\infty$  logarithmic symplectic structures and show our main theorems. In Section 7, we discuss logarithmic deformations. Proofs of our main Theorems are given in Section 8. In Section 9, the unobstructedness theorem can be applied to a complex surface with smooth anti-canonical divisor to obtain unobstructed deformations. We calculate Poisson cohomology groups and the de Rham cohomology groups of the complements. It is intriguing that there are differences between these two cohomology groups when  $D$  has singularities (see Remark 9.4). In Section 9.3, we explain a construction of generalized complex structures  $\mathcal{J}_m$  on  $(3k-1)\mathbb{C}P^3 \# (10k-1)\overline{\mathbb{C}P^2}$  by logarithmic transformations with  $m$  type changing loci. We apply our theorems to generalized complex structures on  $(3k-1)\mathbb{C}P^3 \# (10k-1)\overline{\mathbb{C}P^2}$  to obtain unobstructed deformations of generalized complex structures (see Theorem 9.19).

## 2 Generalized complex structures

Let  $TM$  be the tangent bundle on a differentiable manifold of dimension  $2n$  and  $T^*M$  the cotangent bundle of  $M$ . The symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on the direct sum  $TM \oplus T^*M$  is defined by  $\langle v + \xi, u + \eta \rangle = \frac{1}{2}(\xi(u) + \eta(v))$ , where  $u, v \in TM, \xi, \eta \in T^*M$ . Then the symmetric bilinear form  $\langle \cdot, \cdot \rangle$  yields the fibre bundle  $SO(TM \oplus T^*M)$  with fibre the special orthogonal group. A section of bundle  $SO(TM \oplus T^*M)$  is an endomorphism of  $TM \oplus T^*M$  preserving  $\langle \cdot, \cdot \rangle$  and its determinant is equal to one. If a section  $\mathcal{J}$  of  $SO(TM \oplus T^*M)$  satisfies  $\mathcal{J}^2 = -\text{id}$ , then  $\mathcal{J}$  is called an almost generalized complex structure which gives the decomposition  $(TM \oplus T^*M)^\mathbb{C} = L_{\mathcal{J}} \oplus \overline{L}_{\mathcal{J}}$  into eigenspaces, where  $L_{\mathcal{J}}$  is the eigenspace of eigenvalue  $\sqrt{-1}$  and  $\overline{L}_{\mathcal{J}}$  is the complex conjugate of  $L_{\mathcal{J}}$ . The Courant bracket is defined by

$$[u + \xi, v + \eta]_{\text{co}} = [u, v] + \mathcal{L}_u \eta - \mathcal{L}_v \xi - \frac{1}{2}(di_u \eta - di_v \xi),$$

where  $[u, v]$  denotes the bracket of vector fields  $u$  and  $v$  and  $\mathcal{L}_v \eta$  and  $\mathcal{L}_u \xi$  are the Lie derivatives and  $i_u \eta$  and  $i_v \xi$  stand for the interior products.

If  $L_{\mathcal{J}}$  is closed with respect to the Courant bracket, then  $\mathcal{J}$  is a generalized complex structure, that is,  $[e_1, e_2]_{\text{co}} \in L_{\mathcal{J}}$  for all  $e_1, e_2 \in L_{\mathcal{J}}$ .

The direct sum  $TM \oplus T^*M$  acts on differential forms by the interior and exterior products,

$$e \cdot \phi = (v + \eta) \cdot \phi = i_v \phi + \eta \wedge \phi,$$

where  $e = v + \eta$  and  $v \in TM$  and  $\eta \in T^*M$  and  $\phi$  is a differential forms. Then it turns out that

$$e \cdot e \cdot \phi = (v + \eta) \cdot (v + \eta) \cdot \phi = \eta(v) \phi = \langle e, e \rangle \phi.$$

Since it is the relation of the Clifford algebra with respect to  $\langle \cdot, \cdot \rangle$ , we obtain the action of the Clifford algebra bundle  $\text{Cl}(TM \oplus T^*M)$  on differential forms which is the spin representation.

We define  $\ker \Phi := \{E \in (TM \oplus T^*M)^\mathbb{C} \mid E \cdot \Phi = 0\}$  for a differential form  $\Phi \in \wedge^{\text{even/odd}} T^*M$ . If  $\ker \Phi$  is maximal isotropic, i.e.,  $\dim_{\mathbb{C}} \ker \Phi = 2n$ , then  $\Phi$  is called a *pure spinor* of even/odd type.

A pure spinor  $\Phi$  is *nondegenerate* if  $\ker \Phi \cap \overline{\ker \Phi} = \{0\}$ , i.e.,  $(TM \oplus T^*M)^\mathbb{C} = \ker \Phi \oplus \overline{\ker \Phi}$ . Then a nondegenerate, pure spinor  $\Phi \in \wedge^\bullet T^*M$  gives an almost generalized complex structure  $\mathcal{J}_\Phi$  which satisfies

$$\mathcal{J}_\Phi E = \begin{cases} +\sqrt{-1}E, & E \in \ker \Phi \\ -\sqrt{-1}E, & E \in \overline{\ker \Phi} \end{cases}$$

Conversely, a generalized complex structure  $\mathcal{J}$  arises as  $\mathcal{J}_\Phi$  for a nondegenerate, pure spinor  $\Phi$  which is unique up to multiplication by non-zero functions. Thus there is a one to one correspondence between almost generalized complex structures and non-degenerate, pure spinors modulo multiplication by non-zero functions. The *canonical line bundle*  $K_{\mathcal{J}_\Phi}$  of  $\mathcal{J}_\Phi$  is the complex line bundle generated by the non-degenerate, pure spinor  $\Phi$ .  $\mathcal{J}_\Phi$  is integrable if and only if  $d\Phi = E \cdot \Phi$  for  $E \in (TM \oplus T^*M)^\mathbb{C}$ . The *type number* of  $\mathcal{J} = \mathcal{J}_\Phi$  is a minimal degree of the differential form  $\Phi$ , which is allowed to change on a manifold.

EXAMPLE 2.1. Let  $J$  be the ordinary complex structure on a complex manifold  $X$ . Then the complex structure  $J^*$  on  $T^*X$  is given by  $(J^*\eta)(v) := \eta(Jv)$  for  $\eta \in T^*X, v \in TX$  and we obtain a generalized complex structure  $\mathcal{J}_J$  which is defined by

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix},$$

where the canonical line bundle of  $\mathcal{J}_J$  is the ordinary canonical line bundle  $K_J = \wedge^{n,0}$  which consists of  $n$ -forms. Thus we have Type  $\mathcal{J}_J = n$ .

EXAMPLE 2.2. Let  $\omega$  be a real symplectic structure on a  $2n$ -manifold  $M$ . Then the interior product  $i_v\omega$  of a vector  $v$  yields an isomorphism  $\tilde{\omega} : TM \rightarrow T^*M$  which admits the inverse  $\tilde{\omega}^{-1}$ . Then a generalized complex structure  $\mathcal{J}_\omega$  is defined by

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\tilde{\omega}^{-1} \\ \tilde{\omega} & 0 \end{pmatrix},$$

Then the canonical line bundle of  $\mathcal{J}_\omega$  is generated by

$$\psi = e^{\sqrt{-1}\omega} = 1 + \sqrt{-1}\omega + \frac{1}{2!}(\sqrt{-1}\omega)^2 + \cdots + \frac{1}{n!}(\sqrt{-1}\omega)^n,$$

where the minimal degree of  $\psi$  is 0. Thus we have Type  $\mathcal{J}_\omega = 0$

EXAMPLE 2.3 (The cation of  $b$ -fields). Let  $\mathcal{J}$  be a generalized complex structure which is induced from a non-degenerate, pure spinor  $\phi$ . A  $d$ -closed real 2-form  $b$  acts on  $\phi$  by  $e^b \cdot \phi$  which is also a non-degenerate, pure spinor. Thus  $e^b\phi$  induces a generalized complex structure  $\mathcal{J}_b$ , which is called the action of  $b$ -field on  $\mathcal{J}$ . The generalized complex structure  $\mathcal{J}_b$  gives the decomposition  $TM \oplus T^*M = L_{\mathcal{J}_b} \oplus \overline{L}_{\mathcal{J}_b}$ , where  $L_{\mathcal{J}_b} = \text{Ad}_{e^b} \circ \mathcal{J} \circ \text{Ad}_{e^{-b}}$  and

$$\text{Ad}_{e^b} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

EXAMPLE 2.4 (Poisson deformations). Let  $X$  be a complex manifold and  $\beta$  a holomorphic Poisson structure on a complex manifold  $X$ . Then  $\beta$  gives deformations of new generalized complex structures by  $\mathcal{J}_{\beta t} := \text{Ad}_{e^{\beta t}} \circ \mathcal{J} \circ \text{Ad}_{e^{-\beta}}$  where

$$\text{Ad}_{e^{\beta}} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

The type number is given by Type  $\mathcal{J}_{\beta t_x} = n - 2 \text{ rank of } \beta_x \text{ at } x \in M$ .

### 3 Deformation theory of generalized complex structures

Let  $(M, \mathcal{J})$  be a generalized complex manifold with the decomposition  $(TM \oplus T^*M)^\mathbb{C} = L_{\mathcal{J}} \oplus \overline{L}_{\mathcal{J}}$ . The bundle  $\overline{L}_{\mathcal{J}}$  is a Lie algebroid bundle which yields the Lie algebroid complex,

$$0 \rightarrow \wedge^0 \overline{L}_{\mathcal{J}} \xrightarrow{d_L} \wedge^1 \overline{L}_{\mathcal{J}} \xrightarrow{d_L} \wedge^2 \overline{L}_{\mathcal{J}} \xrightarrow{d_L} \wedge^3 \overline{L}_{\mathcal{J}} \rightarrow \cdots$$

It is known that the Lie algebroid complex is the deformation complex of generalized complex structures. In fact,  $\varepsilon \in \wedge^2 \overline{L}_{\mathcal{J}}$  gives a deformed isotropic subbundle  $L_\varepsilon := \{E + [\varepsilon, E] \mid E \in L_{\mathcal{J}}\}$  which yields a decomposition  $(TM \oplus T^*M)^\mathbb{C} = L_\varepsilon \oplus \overline{L}_\varepsilon$  if  $\varepsilon$  is sufficiently small. The isotropic bundle  $L_\varepsilon$  yields a generalized complex structure if and only if  $\varepsilon$  satisfies the generalized Maurer-Cartan equation

$$d_L \varepsilon + \frac{1}{2}[\varepsilon, \varepsilon]_{\text{Sch}} = 0,$$

where  $[\varepsilon, \varepsilon]_{\text{Sch}}$  denotes the Schouten bracket. The Lie algebroid complex  $(\wedge^\bullet, d_L)$  is an elliptic complex and the spaces of semi-universal deformations (the Kuranishi spaces) of generalized complex structures are constructed [Gu1]. The space of infinitesimal deformations of generalized complex structure is given by the second cohomology group  $H^2(\wedge^\bullet \overline{L}_{\mathcal{J}})$  and the obstruction spaces of generalized complex structure is the third one  $H^3(\wedge^\bullet \overline{L}_{\mathcal{J}})$ .

REMARK 3.1. Let  $\mathcal{J}_b$  be the generalized complex structure by the action of  $d$ -closed  $b$  fields on  $\mathcal{J}$ . Then the isomorphism  $\text{Ad}_{e^b} : \overline{\mathcal{L}}_{\mathcal{J}} \cong \overline{\mathcal{L}}_{\mathcal{J}_b}$  yields the isomorphism of the Lie algebroid complexes  $\wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}} \cong \wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}_b}$ . Thus we have an isomorphism of the Lie algebroid cohomology groups  $H^k(\wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}}) \cong H^k(\wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}_b})$ .

Let  $X = (M, J)$  be a complex manifold and  $\mathcal{J} := \mathcal{J}_J$  denotes the generalized complex structure induced from  $J$  as in Example 2.1. Then it turns out that  $H^\bullet(\wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}})$  is the hypercohomology group of the trivial complex of sheaves:

$$0 \rightarrow \mathcal{O}_X \xrightarrow{0} \Theta \xrightarrow{0} \wedge^2 \Theta \xrightarrow{0} \wedge^3 \Theta \xrightarrow{0} \dots,$$

where 0 denotes the zero map. Thus we have

$$H^k(\wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}_J}) = \oplus_{p+q=k} H^p(X, \wedge^q \Theta),$$

where  $\Theta$  denotes the sheaf of holomorphic vector fields on  $X$  and  $\wedge^q \Theta$  denotes the  $q$ -th skew-symmetric tensor of  $\Theta$ . The infinitesimal deformations is given by

$$H^2(\wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}_J}) = H^2(X, \mathcal{O}_X) \oplus H^1(X, \Theta) \oplus H^0(X, \wedge^2 \Theta),$$

where  $H^1(X, \Theta)$  is the infinitesimal deformations of ordinary complex structures and  $H^2(X, \mathcal{O})$  is given by the action of  $b$ -fields and  $H^0(X, \wedge^2 \Theta)$  corresponds to deformations given by holomorphic Poisson structures.

A holomorphic 2-vector  $\beta \in H^0(X, \wedge^2 \Theta)$  is a holomorphic Poisson structure if  $[\beta, \beta]_{\text{Sch}} = 0$ , where  $[\cdot, \cdot]_{\text{Sch}}$  stands for the Schouten bracket. A holomorphic 2-vector  $\beta$  gives the Poisson bracket of functions by  $\{f, g\}_\beta = \beta(df \wedge dg)$ . Then the Poisson bracket satisfies the Jacobi identity if and only if  $\beta$  is a Poisson structure. A holomorphic Poisson structure  $\beta$  satisfies the generalized Maurer-Cartan equation since  $d_L \beta = \overline{\partial} \beta = 0$ . Thus  $\mathcal{J}_{\beta t} = e^{\beta t} \mathcal{J}_\beta e^{-\beta t}$  gives deformations of generalized complex structures, where  $t$  denotes the complex parameter of deformations. We denote by  $\overline{\mathcal{L}}_{\mathcal{J}_\beta}$  the Lie algebroid bundle of  $\mathcal{J}_\beta$ . A holomorphic Poisson structure  $\beta$  and the Schouten bracket give a map  $\delta_\beta : \wedge^p \Theta \rightarrow \wedge^{p+1} \Theta$  by  $\delta_\beta \alpha := [\beta, \alpha]_{\text{Sch}}$ , where  $\alpha \in \wedge^p \Theta$ . Since the Schouten bracket satisfies the super Jacobi identity and  $[\beta, \alpha]_{\text{Sch}} = [\alpha, \beta]_{\text{Sch}}$ , we have

$$\delta_\beta \circ \delta_\beta(\alpha) = [\beta, [\beta, \alpha]_{\text{Sch}}]_{\text{Sch}} = \frac{1}{2} [\alpha, [\beta, \beta]_{\text{Sch}}]_{\text{Sch}} = 0.$$

Then we obtain the Poisson complex:

$$0 \rightarrow \mathcal{O}_X \xrightarrow{\delta_\beta} \Theta \xrightarrow{\delta_\beta} \wedge^2 \Theta \xrightarrow{\delta_\beta} \wedge^3 \Theta \rightarrow \dots$$

where  $\delta_\beta f = [\beta, f]_{\text{Sch}} = [df, \beta] \in \Theta$  for  $f \in \mathcal{O}_X$  and  $[df, \beta]$  is the commutator in the Clifford algebra which is equal to the coupling between  $df$  and  $\beta$ . A holomorphic Poisson structure  $\beta$  defines a map  $\tilde{\beta}$  from the sheaf of holomorphic 1-forms  $\Omega^1$  to  $\Theta$  by  $[\theta, \beta]$  for  $\theta \in \Omega^1$ . The map  $\tilde{\beta}$  gives the map  $\wedge^p \tilde{\beta} : \Omega^p \rightarrow \wedge^p \Theta$  by

$$\tilde{\beta}(\theta_1 \wedge \dots \wedge \theta_p) = \tilde{\beta}(\theta_1) \wedge \dots \wedge \tilde{\beta}(\theta_p).$$

PROPOSITION 3.2. *The map  $\wedge^p \tilde{\beta}$  induces a map from the holomorphic de Rham complex  $(\Omega^\bullet, d)$  to the Poisson complex  $(\wedge^\bullet \Theta, \delta_\beta)$ ,*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 & \xrightarrow{d} & \dots \\ & & \downarrow \text{id} & & \downarrow \tilde{\beta} & & \downarrow \wedge^2 \tilde{\beta} & & \downarrow \wedge^3 \tilde{\beta} & & \\ 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{\delta_\beta} & \Theta & \xrightarrow{\delta_\beta} & \wedge^2 \Theta & \xrightarrow{\delta_\beta} & \wedge^3 \Theta & \xrightarrow{\delta_\beta} & \dots \end{array}$$

PROOF. It follows from our definitions that  $\tilde{\beta}(df) = [df, \beta] = [\beta, f]_{\text{Sch}} = \delta_\beta f$  and  $\delta_\beta \circ \delta_\beta(f) = \delta_\beta(\tilde{\beta}(df)) = 0$  for  $f \in \mathcal{O}_X$ . Since  $\delta_\beta$  is a derivation, we have

$$\wedge^{p+1} \tilde{\beta} \circ d(fdg_1 \wedge \cdots \wedge dg_p) = \tilde{\beta}(df) \wedge \tilde{\beta}(dg_1) \wedge \cdots \wedge \tilde{\beta}(dg_p) \quad (3.1)$$

$$= \delta_\beta(f \tilde{\beta}(dg_1) \wedge \cdots \wedge \tilde{\beta}(dg_p)) \quad (3.2)$$

$$= \delta_\beta \circ \wedge^p \tilde{\beta}(fdg_1 \wedge \cdots \wedge dg_p), \quad (3.3)$$

where  $g_1, \dots, g_p \in \mathcal{O}_X$ . Thus we obtain  $\wedge^{p+1} \tilde{\beta}(d\gamma) = \delta_\beta \circ \wedge^p \tilde{\beta}(\gamma)$  for  $\gamma \in \wedge^p \Theta$ .  $\square$

The following is already obtained in [LSX]

PROPOSITION 3.3. [LSX] *The hypercohomology of the Poisson complex is isomorphic to the cohomology groups of the Lie algebroid complex of  $\mathcal{J}_\beta$ , i.e.,*

$$\mathbb{H}^k(\wedge^\bullet \Theta) \cong H^k(\wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}_\beta})$$

for all  $k$ .

PROOF. For the completeness of the paper, we shall give a proof of Proposition 3.3. Let  $\wedge_\beta^{0,1}$  be a vector bundle which is the twisted  $\wedge^{0,1}$  by the adjoint action of  $\overline{\beta}$ ,

$$\wedge_\beta^{0,1} := \{e^{\overline{\beta}} \cdot \overline{\theta} \cdot e^{-\overline{\beta}} = \overline{\theta} + [\overline{\beta}, \overline{\theta}] \in \wedge^{1,0} \oplus T^{0,1} M \mid \overline{\theta} \in \wedge^{0,1}\}.$$

We denote by  $\wedge_\beta^{0,q}$  the  $q$ -th skew symmetric tensor of  $\wedge_\beta^{0,1}$ . By the twisted Dolbeault operator  $\overline{\partial}_\beta := e^{\overline{\beta}} \circ \overline{\partial} \circ e^{-\overline{\beta}}$ , we obtain the twisted Dolbeault complex:  $(\wedge_\beta^{0,\bullet}, \overline{\partial}_\beta)$ . Let  $T^{1,0}X$  be the tangent bundle on  $X$  and  $T^{p,0}X$  the skew-symmetric tensor of  $T^{1,0}X$ . Then we also have the complex  $(T^{p,0}X \otimes \wedge_\beta^{0,\bullet}, \overline{\partial}_\beta)$  which are a resolution of  $\wedge^p \Theta$ . As in the Poisson complex, the map  $\delta_\beta$  defines the map between complexes  $\delta_\beta : (T^{p,0}X \otimes \wedge_\beta^{0,\bullet}, \overline{\partial}_\beta)$  to  $(T^{p+1,0}X \otimes \wedge_\beta^{0,\bullet}, \overline{\partial}_\beta)$ . Thus we have a double complex  $(T^{\bullet,0}X \otimes \wedge_\beta^{0,\bullet}, \delta_\beta, \overline{\partial}_\beta)$ . It turns out that the total complex of the double complex is the Lie algebroid complex  $(\wedge^\bullet, \overline{\mathcal{L}}_{\mathcal{J}_\beta}, d_L)$ . Since the cohomology groups of the total complex is the hypercohomology groups of the Poisson complex, we have  $\mathbb{H}^k(\wedge^\bullet \Theta) \cong H^k(\wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}_\beta})$ .  $\square$

## 4 Log symplectic structure $\omega_{\mathbb{C}}$ in holomorphic category and generalized complex structure $\mathcal{J}_\phi$

Let  $X$  be a complex manifold of complex dimension  $n$ . Let  $D$  be a smooth divisor on  $X$  which admits holomorphic coordinates  $(z_1, \dots, z_{2m})$  such that  $D$  is given by  $z_1 = 0$ . We call such coordinates  $(z_1, \dots, z_{2n})$  logarithmic coordinates. A *logarithmic symplectic structure*<sup>†</sup>  $\omega_{\mathbb{C}}$  is a  $d$ -closed, logarithmic 2-form on  $X$  which satisfies the followings (1) and (2) :

(1) There exist logarithmic coordinates  $(z_1, \dots, z_{2m})$  on a neighborhood of every point in  $D$  such that  $\omega_{\mathbb{C}}$  is written as

$$\omega_{\mathbb{C}} = \frac{dz_1}{z_1} \wedge dz_2 + dz_3 \wedge dz_4 + \cdots + dz_{2m-1} \wedge dz_{2m}. \quad (4.1)$$

(2) On the complement  $X \setminus D$ ,  $\omega_{\mathbb{C}}$  is a holomorphic symplectic form.

Then  $\phi := e^{\omega_{\mathbb{C}}}$  is a  $d$ -closed, non-degenerate, pure spinor on  $X \setminus D$  which induces the generalized complex structure  $\mathcal{J}_\phi$  on  $X \setminus D$ . On a neighborhood of  $D$ ,  $z_1 e^{\omega_{\mathbb{C}}}$  is also a nondegenerate, pure spinor. Hence  $\mathcal{J}_\phi$  can be extended as a generalized complex structure on  $X$ . The type number of  $\mathcal{J}_\phi$  is given by the followings: On the divisor  $D = \{z_1 = 0\}$ ,  $\mathcal{J}_\phi$  is induced from  $z_1 \phi$ , where

$$z_1 e^{\omega_{\mathbb{C}}} \big|_{z_1=0} = dz_1 \wedge dz_2 + \cdots = (dz_1 \wedge dz_2) \wedge e^{\tilde{\omega}_{\mathbb{C}}},$$

<sup>†</sup>The notion of logarithmic symplectic structures was introduced in [Go3].

where  $\tilde{\omega}_{\mathbb{C}} = dz_3 \wedge dz_4 + \cdots + dz_{2m-1} \wedge dz_{2m}$ . Thus the minimal degree of  $z_1\phi$  on  $D$  is equal to 2. On the complement  $X \setminus D$ ,  $\mathcal{J}_{\phi}$  is given by  $\phi$  whose the minimal degree is 0. Thus we have

$$\text{Type } \mathcal{J}_{\phi}(x) = \begin{cases} 2 & (x \in D) \\ 0 & (x \notin D) \end{cases}$$

## 5 Lie algebroid cohomology groups of $\mathcal{J}_{\phi}$ and logarithmic Poisson structure $\beta$

Let  $\omega_{\mathbb{C}}$  be a logarithmic symplectic structure on a complex manifold  $X$ . Then  $\omega_{\mathbb{C}}$  gives the isomorphism between the sheaf of holomorphic logarithmic tangent vectors  $\Theta(-\log D)$  and the sheaf of logarithmic 1-forms  $\Omega^1(\log D)$  which also induces the isomorphism  $\wedge^2 \Theta(-\log D) \cong \Omega^2(\log D)$ . Then  $\omega_{\mathbb{C}} \in H^0(X, \Omega^2(\log D))$  admits the dual 2-vector  $\beta \in H^0(X, \wedge^2 \Theta(-\log D))$ . Since  $\omega_{\mathbb{C}}$  is  $d$ -closed,  $\beta$  is a Poisson structure. We call the  $\beta$  a *holomorphic log Poisson structure*.

The interior product  $i_v \omega_{\mathbb{C}}$  by a holomorphic vector field  $v$  of  $\omega_{\mathbb{C}}$  gives a meromorphic 1-form with simple pole along  $D$ . Thus  $\omega_{\mathbb{C}}$  yields a map  $\tilde{\omega}_{\mathbb{C}}$  from  $\Theta$  to  $\hat{\Omega}^1$ , where  $\hat{\Omega}^1$  is defined to be the image of  $\tilde{\omega}_{\mathbb{C}}$ . Then the map  $\tilde{\omega}_{\mathbb{C}}$  is the inverse of the map  $\tilde{\beta} : \hat{\Omega}^1 \rightarrow \Theta$  as in Section 3. We define  $\hat{\Omega}^p$  by  $\hat{\Omega}^p = \wedge^p \hat{\Omega}^1$ . Then the exterior derivative  $d$  gives a complex  $(\hat{\Omega}^{\bullet}, d)$ . Since  $\wedge^p \tilde{\omega}_{\mathbb{C}}$  gives an isomorphism  $\wedge^p \Theta \cong \hat{\Omega}^p$  which is the inverse of  $\wedge^p \tilde{\beta}$ , it follows from Proposition 3.2 that the complex  $(\wedge^{\bullet} \hat{\Omega}, d)$  is isomorphic to the Poisson complex  $(\wedge^{\bullet} \Theta, \delta_{\beta})$  and we obtain  $\mathbb{H}^k(\wedge^{\bullet} \Theta) \cong \mathbb{H}^k(\hat{\Omega}^{\bullet})$ . We denote by  $(\Omega^{\bullet}(\log D), d)$  the holomorphic log complex.

Since the log complex  $(\Omega^{\bullet}(\log D), d)$  is a subcomplex of  $(\hat{\Omega}^{\bullet}, d)$ , we have the short exact sequence of complexes:

$$0 \rightarrow \Omega^{\bullet}(\log D) \rightarrow \hat{\Omega}^{\bullet} \rightarrow Q^{\bullet} \rightarrow 0,$$

where  $Q^{\bullet}$  denotes the quotient complex.

LEMMA 5.1. *Let  $\mathcal{H}^{\bullet}(Q^{\bullet})$  be the cohomology sheaves of the complex  $Q^{\bullet}$ . Then we have that  $\mathcal{H}^{\bullet}(Q^{\bullet}) = \{0\}$ .*

PROOF. Let  $(z_1, \dots, z_{2n})$  be logarithmic coordinates of a neighborhood of  $x \in D$  such that  $\omega_{\mathbb{C}}$  is given by (4.1). Then we see that

$$i_{\frac{\partial}{\partial z_1}} \omega_{\mathbb{C}} = \frac{dz_2}{z_1}, \quad i_{\frac{\partial}{\partial z_2}} \omega_{\mathbb{C}} = -\frac{dz_1}{z_1}$$

It follows that the germ of the image  $\hat{\Omega}_x^1$  is generated by

$$\left\langle \frac{dz_2}{z_1}, \frac{dz_1}{z_1}, dz_3, \dots, dz_{2m} \right\rangle$$

over  $\mathcal{O}_{X,x}$ . Then we see that the germ of  $\hat{\Omega}_x^2$  is generated by

$$\left\langle \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_1}, \frac{dz_1}{z_1} \wedge dz_i, \frac{dz_2}{z_1} \wedge dz_j, dz_i \wedge dz_j \mid i, j = 1, \dots, 2m \right\rangle$$

over  $\mathcal{O}_{X,x}$ . Then it turns out that every  $\alpha \in \hat{\Omega}_x^2$  is written as

$$\alpha = \alpha_0 \wedge \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_1} + \alpha_1 \wedge \frac{dz_1}{z_1} + \alpha_2 \wedge \frac{dz_2}{z_1} + \alpha_3$$

where  $\alpha_0, \alpha_1, \alpha_2, \alpha_3$  are holomorphic forms. Since  $d(\frac{dz_2}{z_1}) = -\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_1}$ , it follows that

$$\gamma := \alpha + (-1)^{|\alpha_0|} d(\alpha_0 \wedge \frac{dz_2}{z_1}) \in \hat{\Omega}_x^2$$

is a meromorphic form with pole of order at most one on  $D$ . We assume that  $d\alpha$  is a logarithmic form. Then  $z_1 d\alpha$  is holomorphic and  $z_1 d\gamma$  is also holomorphic. Since  $z_1 \gamma$  is holomorphic, it follow

that  $\gamma$  is a logarithmic form. If  $\alpha$  is a representative of the germ of the cohomology sheaves  $\mathcal{H}^\bullet(Q^\bullet)_x$ , then  $[\alpha] = [\gamma] \in \mathcal{H}^\bullet(Q^\bullet)_x$  and  $d\alpha$  is a logarithmic form. Thus it follows that  $[\alpha] = [\gamma] = 0 \in \mathcal{H}^\bullet(Q^\bullet)_x$  since  $\gamma$  is a logarithmic form and  $\gamma \equiv 0$  in  $Q^\bullet$ .  $\square$

PROPOSITION 5.2. *The complex  $(\wedge^\bullet \hat{\Omega}, d)$  is quasi-isomorphic to the logarithmic complex  $(\Omega^\bullet(\log D), d)$ . Thus the cohomology groups  $\mathbb{H}^k(\wedge^\bullet \Theta) \cong \mathbb{H}^k(\wedge^\bullet \hat{\Omega}) \cong H^k(\wedge^\bullet \bar{L}_\phi)$  are given by  $H^k(X \setminus D, \mathbb{C})$ .*

PROOF. It follows from Lemma 5.1 that the map  $(\Omega^\bullet(\log D), d) \rightarrow (\hat{\Omega}^\bullet, d)$  is a quasi-isomorphism. Thus we have  $\mathbb{H}^k(\Omega^\bullet(\log D)) \cong \mathbb{H}^k(\hat{\Omega}^\bullet)$ . It is known that the hypercohomology groups  $\mathbb{H}^k(\wedge^\bullet \Omega(\log D))$  of the log complex are the cohomology groups of the complement  $H^k(X \setminus D, \mathbb{C})$ . It follows from Proposition 3.3 that  $H^k(\wedge^\bullet \bar{L}_\phi) \cong \mathbb{H}^k(\wedge^\bullet \Theta)$ . Thus we obtain  $H^k(\wedge^\bullet \bar{L}_\phi) \cong H^k(X \setminus D, \mathbb{C})$ .  $\square$

## 6 Unobstructed deformations of generalized complex structures induced from $C^\infty$ logarithmic symplectic structures

Let  $M$  be a differentiable manifold of real dimension  $4m$  and  $D$  a submanifold of real codimension 2. We assume that there is an open cover  $M = \cup_\alpha U_\alpha$  such that each  $U_\alpha$  is an open set of  $\mathbb{C}^{2m}$  with complex coordinates  $(z_\alpha^1, \dots, z_\alpha^{2m})$  and  $D$  is locally given by  $\{z_\alpha^1 = 0\}$  for  $U_\alpha \cap D \neq \emptyset$ . We say  $(z_\alpha^1, \dots, z_\alpha^{2m})$  logarithmic coordinates of  $D$ . Note that we do not assume that  $M$  is a complex manifold. In fact, defining equations of  $D$  satisfies  $z_\alpha^1 = e^{f_{\alpha,\beta}} z_\beta^1$  on  $U_\alpha \cap U_\beta$ , where  $f_{\alpha,\beta}$  is a smooth function on  $U_\alpha \cap U_\beta$ .

DEFINITION 6.1. A  $C^\infty$  logarithmic symplectic structure<sup>†</sup>  $\omega_{\mathbb{C}}$  is a  $d$ -closed complex 2-form which satisfies the followings:

(1) On a neighborhood of  $D$ ,  $\omega_{\mathbb{C}}$  is locally given by

$$\omega_{\mathbb{C}} = \frac{dz_1}{z_1} \wedge dz_2 + dz_3 \wedge dz_4 + \dots + dz_{2m-1} \wedge dz_{2m},$$

where  $(z_1, \dots, z_{2m})$  are logarithmic coordinates of  $D = \{z_1 = 0\}$ .

(2) On a neighborhood of the complement  $M \setminus D$ ,  $\omega_{\mathbb{C}} = b + \sqrt{-1}\omega$  where  $b$  is a  $d$ -closed 2-form and  $\omega$  denotes a real symplectic structure.

Then  $\phi = e^{\omega_{\mathbb{C}}}$  is a  $d$ -closed, non-degenerate, pure spinor which induces the generalized complex structure  $\mathcal{J}_\phi$  on the complement  $M \setminus D$ . In fact,  $z_1 \phi$  is a non-degenerate pure spinor on a neighborhood  $U$  of  $D$ . Thus it follows that  $\phi := e^{\omega_{\mathbb{C}}}$  defines a generalized complex structure  $\mathcal{J}_\phi$  on  $M$ .

Then we have the following theorem :

THEOREM 6.2. *Let  $\omega_{\mathbb{C}}$  be a  $C^\infty$  logarithmic symplectic structure on  $M$  and  $\mathcal{J}_\phi$  the generalized complex structure which is induced from  $\phi = e^{\omega_{\mathbb{C}}}$ . Then the Lie algebroid cohomology groups  $H^k(\wedge^\bullet \bar{L}_\phi)$  of  $\mathcal{J}_\phi$  is isomorphic to  $H^k(M \setminus D, \mathbb{C})$ .*

THEOREM 6.3. *Let  $\omega_{\mathbb{C}}$  be a  $C^\infty$  logarithmic symplectic structure on  $M$  and  $\mathcal{J}_\phi$  the generalized complex structure which is induced from  $\phi = e^{\omega_{\mathbb{C}}}$ . Then deformations of  $\mathcal{J}_\phi$  are unobstructed and the space of infinitesimal deformations is given by  $H^2(M \setminus D, \mathbb{C})$ .*

In order to prove our theorems, we shall introduce  $C^\infty$  logarithmic deformations of  $\mathcal{J}_\phi$  and the  $C^\infty$  logarithmic deformations are unobstructed in next Section.

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<sup>†</sup> Note that the notion of  $C^\infty$  logarithmic symplectic structures is different from the one of singular symplectic structures as in [GMP, GuLi] whose singular loci are real codimension 1.



## 7 Logarithmic deformations of $\mathcal{J}_\phi$

A  $C^\infty$  logarithmic vector field  $V$  on a manifold  $M$  along  $D$  is a  $C^\infty$  vector field which is locally given by

$$V = f_1 z_1 \frac{\partial}{\partial z_1} + g_1 \frac{\partial}{\partial \bar{z}_1} + \sum_{i=2}^{2m} f_i \frac{\partial}{\partial z_i} + g_i \frac{\partial}{\partial \bar{z}_i},$$

where  $f_i, g_i$  ( $i = 1, \dots, 2m$ ) are  $C^\infty$  functions and  $(z_1, \dots, z_{2m})$  are logarithmic coordinates of  $D = \{z_1 = 0\}$ . Thus a  $C^\infty$  logarithmic vector field  $V$  preserves the ideal  $(z_1)$  which is an analog of the notion of logarithmic vector fields in complex geometry. We denote by  $\mathcal{T}_{\log} M$  the sheaf of  $C^\infty$  logarithmic vector fields. The sheaf  $\mathcal{T}_{\log} M$  is locally free which gives a  $C^\infty$  vector bundle  $T_{\log} M$ . Our generalized complex structure  $\mathcal{J}_\phi$  gives the decomposition  $TM \oplus T^*M = L_\phi \oplus \bar{L}_\phi$  and the Lie algebroid complex  $(\wedge^\bullet \bar{L}_\phi, d_{L_\phi})$ . We define a subbundle  $\bar{L}_\phi^{\log}$  by

$$\bar{L}_\phi^{\log} = \bar{L}_\phi \cap (T_{\log} M \oplus T^*M)^\mathbb{C}$$

Then we obtain the subcomplex  $(\wedge^\bullet \bar{L}_\phi^{\log}, d_L)$  of the Lie algebroid complex  $(\wedge^\bullet \bar{L}_\phi, d_L)$ . Then we have

PROPOSITION 7.1. *The cohomology group  $H^k(\wedge^\bullet \bar{L}_\phi^{\log})$  of the subcomplex  $(\wedge^\bullet \bar{L}_\phi^{\log}, d_L)$  is isomorphic to the cohomology group  $H^k(M \setminus D, \mathbb{C})$*

PROOF. Let  $\bar{\mathcal{L}}_\phi^{\log}$  be the sheaf of  $C^\infty$  sections of the bundle  $\bar{L}_\phi^{\log}$  and  $\wedge^p \bar{\mathcal{L}}_\phi^{\log}$  the  $p$ -th skew symmetric tensor of  $\bar{\mathcal{L}}_\phi^{\log}$ . Then we have the complex of sheaves:

$$0 \longrightarrow \wedge^0 \bar{\mathcal{L}}_\phi^{\log} \xrightarrow{d_L} \wedge^1 \bar{\mathcal{L}}_\phi^{\log} \xrightarrow{d_L} \wedge^2 \bar{\mathcal{L}}_\phi^{\log} \xrightarrow{d_L} \dots \quad (7.1)$$

Since  $\wedge^p \bar{\mathcal{L}}_\phi^{\log}$  is a soft sheaf, the hypercohomology groups  $\mathbb{H}^k(\wedge^\bullet \bar{\mathcal{L}}_\phi^{\log})$  of the complex of sheaves  $(\wedge^\bullet \bar{\mathcal{L}}_\phi^{\log}, d_L)$  is given by global sections and we have  $\mathbb{H}^k(\wedge^\bullet \bar{\mathcal{L}}_\phi^{\log}) \cong H^k(\wedge^\bullet \bar{L}_\phi^{\log})$ . The interior product  $i_v \omega_\mathbb{C}$  of a vector  $v$  by  $\omega_\mathbb{C}$  restricted to the complement  $M \setminus D$  gives a map from  $T_{\log} M$  to 1-forms on  $M \setminus D$  which induces a map  $\tilde{\omega}_\mathbb{C}$  from  $\bar{\mathcal{L}}_\phi^{\log}$  to the sheaf of differential 1-forms on  $M \setminus D$  by  $\tilde{\omega}_\mathbb{C}(v + \theta) = -i_v \omega_\mathbb{C} + \theta$ . Then we have the map  $\wedge^p \tilde{\omega}_\mathbb{C} : \wedge^p \bar{\mathcal{L}}_\phi^{\log} \rightarrow \mathcal{A}^p(M \setminus D)$  by

$$\wedge^p \tilde{\omega}_\mathbb{C}(v_1 \wedge \dots \wedge v_s \wedge \theta_1 \wedge \dots \wedge \theta_s) = (-1)^s i_{v_1} \omega_\mathbb{C} \wedge \dots \wedge i_{v_s} \omega_\mathbb{C} \wedge \theta_1 \wedge \dots \wedge \theta_s, \quad (7.2)$$

where  $s + t = p$  and  $\mathcal{A}^p(M \setminus D)$  denotes the sheaf of  $p$ -forms on  $M \setminus D$ . The map  $\wedge^p \tilde{\omega}_\mathbb{C}$  gives the map  $\wedge^\bullet \tilde{\omega}_\mathbb{C}$  from the complex  $(\wedge^\bullet \bar{\mathcal{L}}_\phi^{\log}, d_L)$  to the de Rham complex  $(\mathcal{A}^\bullet(M \setminus D), d)$ . We shall show that the map  $\wedge^\bullet \tilde{\omega}_\mathbb{C} : (\wedge^\bullet \bar{\mathcal{L}}_\phi^{\log}, d_L) \rightarrow (\mathcal{A}^\bullet(M \setminus D), d)$  is quasi-isomorphic. In order to obtain the quasi-isomorphism, we shall determine the cohomology sheaves  $\mathcal{H}^k(\wedge^\bullet \bar{\mathcal{L}}_\phi^{\log})(U)$  of the complex  $(\wedge^\bullet \bar{\mathcal{L}}_\phi^{\log}, d_L)$  restricted to a neighborhood  $U$  in the followings two cases :

(1) If  $U$  is a neighborhood of  $D$  with logarithmic coordinates  $(z_1, \dots, z_{2m})$ , then the logarithmic coordinates define the complex structure on  $U$  such that  $\omega_\mathbb{C}|_U$  is a logarithmic symplectic structure which is the dual of holomorphic logarithmic Poisson structure  $\beta$  as in Section 5. Then it turns out that the cohomology groups  $\mathcal{H}^k(\wedge^\bullet \bar{\mathcal{L}}_\phi^{\log})(U)$  is given by the hypercohomology groups  $\mathbb{H}^k(\wedge^\bullet(\Theta(-\log D)))$  of the Poisson complex of multi-logarithmic tangent vectors:

$$0 \rightarrow \wedge^0 \Theta(-\log D) \xrightarrow{\delta_\beta} \wedge^1 \Theta(-\log D) \xrightarrow{\delta_\beta} \wedge^2 \Theta(-\log D) \xrightarrow{\delta_\beta} \dots$$

The map  $\wedge^\bullet \tilde{\omega}_\mathbb{C}$  restricted to  $U$  gives an isomorphism from the complex of logarithmic multi-tangent vectors  $(\wedge^\bullet \Theta(-\log D), \delta_\beta)$  to the complex of logarithmic forms  $(\Omega^\bullet(\log D), d)$  which induces the isomorphism between cohomology groups  $\mathbb{H}^k(\Omega^\bullet(\log D)) \cong \mathbb{H}^k(\wedge^\bullet \Theta(-\log D))$ . It is known that the hypercohomology groups  $\mathbb{H}^k(\Omega^\bullet(\log D))$  of the complex of logarithmic forms is  $H^k(U \setminus U \cap D, \mathbb{C})$ . Thus we have  $\mathcal{H}^k(\wedge^\bullet \bar{\mathcal{L}}_\phi^{\log})(U) \cong H^k(U \setminus U \cap D, \mathbb{C})$ .

(2) If  $U$  is a neighborhood of the complement  $M \setminus D$ , then  $\omega_\mathbb{C}|_U$  is given by  $b + \sqrt{-1}\omega$ . It follows that the map  $\wedge^\bullet \tilde{\omega}_\mathbb{C} : (\wedge^\bullet \bar{\mathcal{L}}_\phi^{\log}(U), d_L) \rightarrow (\mathcal{A}^\bullet(U), d)$  is an isomorphism and  $\mathcal{H}^k(\wedge^\bullet \bar{\mathcal{L}}_\phi^{\log})(U) \cong H^k(U, \mathbb{C})$ .

Then it follows that the map  $\wedge^\bullet \tilde{\omega}_\mathbb{C} : (\wedge^\bullet \bar{\mathcal{L}}_\phi^{\log}, d_L) \rightarrow (\mathcal{A}^\bullet(M \setminus D), d)$  is a quasi-isomorphism. Hence we obtain  $H^k(\wedge^\bullet \bar{L}_\phi^{\log}) \cong H^k(M \setminus D, \mathbb{C})$ .  $\square$

PROPOSITION 7.2. *The second cohomology group  $H^2(\wedge^\bullet \overline{\mathcal{L}}_\phi^{\log}) \cong H^2(M \setminus D, \mathbb{C})$  gives unobstructed deformations of generalized complex structures.*

PROOF. A  $C^\infty$  logarithmic 1-form  $\theta$  is a  $C^\infty$  1-form on  $M \setminus D$  which is written on a neighborhood  $U$  of  $D$  by

$$\theta = f_1 \frac{dz_1}{z_1} + g_1 d\bar{z}_1 + \sum_{i=2}^{2m} f_i dz_i + g_i d\bar{z}_i,$$

where  $f_i, g_i$  are  $C^\infty$  functions and  $(z_1, \dots, z_{2m})$  denotes logarithmic coordinates. Let  $\mathcal{T}_{\log}^* M$  be the sheaf of  $C^\infty$  sections of  $C^\infty$  logarithmic 1-forms and  $\wedge^p \mathcal{T}_{\log}^* M$  the  $p$ -th skew symmetric tensors of  $\mathcal{T}_{\log}^* M$ . Then we have the complex of  $C^\infty$  logarithmic forms:

$$0 \rightarrow \wedge^0 \mathcal{T}_{\log}^* M \xrightarrow{d} \wedge^1 \mathcal{T}_{\log}^* M \xrightarrow{d} \wedge^2 \mathcal{T}_{\log}^* M \xrightarrow{d} \dots$$

It turns out that the hypercohomology groups  $\mathbb{H}^k(\wedge^\bullet \mathcal{T}_{\log}^* M)$  of the complex of  $C^\infty$  logarithmic forms is  $H^k(M \setminus D, \mathbb{C})$ . Thus every element of  $H^2(M \setminus D, \mathbb{C})$  admits a  $d$ -closed representative  $\alpha \in C^\infty(M, \wedge^2 \mathcal{T}_{\log}^* M)$ . If  $\alpha$  is sufficiently small, then  $e^{\omega_{\mathbb{C}} + \alpha}$  is a  $d$ -closed, non-degenerate, pure spinor on  $M \setminus D$  which induces a family of deformations of generalized complex structures on  $M \setminus D$  parametrized by an open set of  $H^2(M \setminus D, \mathbb{C})$ . On a neighborhood of  $D$  with logarithmic coordinates  $(z_1, \dots, z_{2m})$ ,  $\omega_{\mathbb{C}}$  is given by  $\omega_{\mathbb{C}} = \frac{dz_1}{z_1} \wedge dz_2 + \hat{\omega}_{\mathbb{C}}$  and  $\alpha$  is written as  $\alpha = \frac{dz_1}{z_1} \wedge \alpha_1 + \gamma$ , where  $\hat{\omega}_{\mathbb{C}}, \alpha_1, \gamma$  are  $C^\infty$  2-forms. Then we have  $\omega_{\mathbb{C}} + \alpha = \frac{dz_1}{z_1} \wedge (dz_2 + \alpha_1) + \hat{\omega}_{\mathbb{C}} + \gamma$ . Then  $z_1 e^{\omega_{\mathbb{C}} + \alpha}$  restricted to  $D = \{z_1 = 0\}$  is given by

$$z_1 e^{\omega_{\mathbb{C}} + \alpha}|_{z_1=0} = dz_1 \wedge (dz_2 + \alpha_1) \wedge e^{\hat{\omega}_{\mathbb{C}} + \gamma}$$

Hence  $z_1 e^{\omega_{\mathbb{C}} + \alpha}|_{z_1=0}$  is also a non-degenerate, pure spinor on  $D$  for sufficiently small  $\alpha$ . Thus  $e^{\omega_{\mathbb{C}} + \alpha}$  gives deformations of generalized complex structures on  $M$  which are parametrized by an open set  $H^2(M \setminus D, \mathbb{C})$ .  $\square$

## 8 Proof of main theorems

PROOF OF THEOREM 6.2. Let  $\overline{\mathcal{L}}_\phi$  be the sheaf of germs of smooth sections of the bundle  $\overline{\mathcal{L}}_\phi$ . Then the Lie algebroid complex gives the complex of sheaves:

$$0 \rightarrow \wedge^0 \overline{\mathcal{L}}_\phi \xrightarrow{d_L} \wedge^1 \overline{\mathcal{L}}_\phi \xrightarrow{d_L} \wedge^2 \overline{\mathcal{L}}_\phi \xrightarrow{d_L} \dots \quad (8.1)$$

The hypercohomology groups of the complex of sheaves are isomorphic to the cohomology groups of the Lie algebroid complex since  $\wedge^p \overline{\mathcal{L}}_\phi$  are a soft sheaf. We shall apply the similar argument as in Proposition 7.1 to the complex (8.1). The interior product  $i_v \omega_{\mathbb{C}}$  of a vector field  $v$  by  $\omega_{\mathbb{C}}$  restricted to  $M \setminus D$  gives the map  $\wedge^p \tilde{\omega}_{\mathbb{C}} : \wedge^p \overline{\mathcal{L}}_\phi \rightarrow \mathcal{A}^p(M \setminus D)$  as in (7.2) which yields a map  $\wedge^\bullet \tilde{\omega}_{\mathbb{C}}$  from the complex of sheaves  $(\wedge^\bullet \overline{\mathcal{L}}_\phi, d_L)$  to the de Rham complex  $(\mathcal{A}^\bullet(M \setminus D), d)$ . We shall show the map  $\wedge^\bullet \tilde{\omega}_{\mathbb{C}}$  is a quasi-isomorphism.

The sheaves  $\mathcal{H}^\bullet(\wedge^\bullet \overline{\mathcal{L}}_\phi)$  of cohomology groups of (8.1) are determined by the following two cases (1) and (2):

(1) If  $U$  is a neighborhood of  $D$  admitting logarithmic coordinates of  $D$ , then the logarithmic coordinates define the complex structure on  $U$  such that  $\omega_{\mathbb{C}}|_U$  is a logarithmic symplectic structure which is the dual of holomorphic logarithmic Poisson structure  $\beta$  as in Section 5. It follows from Proposition 5.2 that the cohomology  $\mathcal{H}^k(\wedge^\bullet \overline{\mathcal{L}}_\phi)(U)$  is isomorphic to  $H^k(U \setminus U \cap D, \mathbb{C})$ . Thus if  $x \in M \setminus D$ , then  $\wedge^\bullet \tilde{\omega}_{\mathbb{C}}$  induces an isomorphism

$$\mathcal{H}^k(\wedge^\bullet \overline{\mathcal{L}}_\phi)_x \cong \lim_{x \in U} H^k(U)$$

(2) If  $U$  is a neighborhood of the complement  $M \setminus D$ , then  $\omega_{\mathbb{C}} = b + \sqrt{-1}\omega$  gives an isomorphism from the complex  $(\wedge^\bullet \overline{\mathcal{L}}_\phi(U), d_L)$  to the de Rham complex  $(\mathcal{A}^\bullet(U), d)$ .

It follows that the complex of sheaves  $\wedge^\bullet \mathcal{L}_{\mathcal{J}}$  is quasi-isomorphic to the complex of sheaves  $(\mathcal{A}^\bullet(M \setminus D), d)$ . Thus the hypercohomology groups of the complex  $\wedge^\bullet \mathcal{L}_{\mathcal{J}}$  are  $H^\bullet(M \setminus D)$ . Hence the cohomology groups of the Lie algebroid complex are also  $H^\bullet(M \setminus D)$ .  $\square$

PROOF OF THEOREM 6.3. From Theorem 6.2, we obtain  $H^2(\wedge^\bullet \overline{\mathcal{L}}_\phi) \cong H^2(M \setminus D, \mathbb{C})$ . It follows from Proposition 7.2 that we already obtain deformations of generalized complex structures on  $M$  parametrized by an open set  $H^2(M \setminus D, \mathbb{C})$ . Thus we obtain unobstructed deformations of generalized complex structures.  $\square$

## 9 Generalized complex structures on 4-manifolds

### 9.1 Non-degenerate, pure spinors of even type on 4-manifolds

Let  $M$  be a 4-dimensional manifold. Then the spinor inner metric of even forms is defined by

$$\langle \phi, \psi \rangle := \phi_0 \psi_4 - \phi_2 \wedge \psi_2 + \phi_4 \psi_0 \in \wedge^4 T^* M,$$

where  $\phi = \phi_0 + \phi_2 + \phi_4, \psi = \psi_0 + \psi_2 + \psi_4 \in \wedge^* T^* M$  and  $\phi_i, \psi_i \in \wedge^i T^* M$ . Then the spinor inner metric gives a simple description of non-degenerate, pure spinors of even type, that is,  $\phi \in \wedge^{\text{even}} T^* M$  is a non-degenerate, pure spinor if and only if  $\phi$  satisfies the followings:

$$\langle \phi, \phi \rangle := 2\phi_0 \wedge \phi_4 - \phi_2 \wedge \phi_2 = 0, \quad (9.1)$$

$$\langle \phi, \overline{\phi} \rangle := \phi_0 \wedge \overline{\phi}_4 + \phi_4 \wedge \overline{\phi}_0 - \phi_2 \wedge \overline{\phi}_2 \neq 0 \quad (9.2)$$

A non-degenerate, pure spinor  $\phi$  gives a generalized complex structure if and only if

$$d\phi = e \cdot \phi, \quad (9.3)$$

where  $e = v + \theta \in TM \oplus T^*M$ .

EXAMPLE 9.1. Let  $(z_1, z_2)$  be the coordinates of  $\mathbb{C}^2$  and  $\phi = 1 + \frac{dz_1}{z_1} \wedge dz_2$ . Then  $z_1 \phi = z_1 + dz_1 \wedge dz_2$  is a non-degenerate, pure spinor which induces the generalized complex structure  $\mathcal{J}_\phi$ . On the divisor  $D := \{z_1 = 0\}$ , we have  $\phi|_D = dz_1 \wedge dz_2$  and  $\mathcal{J}_\phi|_D$  is induced from a complex structure and Type  $\mathcal{J}_\phi|_D = 2$ . On the complement  $\mathbb{C}^2 \setminus D$ ,  $\phi = \exp(\frac{dz_1}{z_1} \wedge dz_2) = \exp(b + \sqrt{-1}\omega)$ , where  $b$  is a  $d$ -closed real 2-form and  $\omega$  is a symplectic form. Thus  $\mathcal{J}_\phi$  is coming from a symplectic structure twisted by the action of  $b$ -field and Type  $\mathcal{J}_\phi|_{\mathbb{C}^2 \setminus D} = 0$

### 9.2 Unobstructed deformations of generalized complex structures on Poisson surfaces

Let  $S = (M, J)$  be a complex surface with effective anti-canonical line bundle  $K^{-1}$  and  $\mathcal{J}_J$  the generalized complex structure given by the complex structure  $J$  as in Example 2.1. Then a section  $\beta \in K^{-1}$  is a holomorphic Poisson structure on  $S$  which gives Poisson deformations  $\{\mathcal{J}_{\beta t}\}$  of generalized complex structures as in Example 2.4. Then we have

THEOREM 9.2. *If the zero set of  $\beta$  is a smooth divisor  $D$  of  $S = (M, J)$ , then the Lie algebroid cohomology groups  $H^k(\wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}_\beta})$  is isomorphic to  $H^k(M \setminus D, \mathbb{C})$  and deformations of generalized complex structure  $\mathcal{J}_\beta$  are parametrized by an open set  $H^2(M \setminus D, \mathbb{C})$ .*

PROOF. Let  $\omega_{\mathbb{C}}$  be the dual of  $\beta$  which is a logarithmic symplectic structure on  $S$  of  $D$ . Since  $\mathcal{J}_\beta$  is induced from  $\phi = e^{\omega_{\mathbb{C}}}$ , the results follows from Theorem 6.2 and Theorem 6.3.  $\square$

In general, if an anti-canonical divisor  $D$  on a complex surface  $S$  is not smooth, the Lie algebroid cohomology groups (the hypercohomology groups of Poisson complex) are different from the singular cohomology groups of the complement  $M \setminus D$ . We assume that  $D$  is given by the zero locus of a Poisson structure  $\beta$  which has isolated singular points  $\{p_i\}_{i=1}^m$  and  $\beta$  is written as  $f_i \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$  at a neighborhood of each  $p_i$ , where  $f_i \in \mathcal{O}_{S, p_i}$ . Then we have the complex  $(\hat{\Omega}^\bullet, d)$  which is isomorphic to the Poisson complex  $(\wedge^\bullet \Theta, \delta_\beta)$  as in Section 5. The log complex  $\Omega^\bullet(\log D)$  is a subcomplex of  $\hat{\Omega}^\bullet$  and we have the short exact sequence of complexes

$$0 \rightarrow \Omega^\bullet(\log D) \rightarrow \hat{\Omega}^\bullet \rightarrow Q^\bullet \rightarrow 0 \quad (9.4)$$

The cohomology sheaves of the complex  $Q^\bullet$  are given by

$$\mathcal{H}^i(Q^\bullet) = \begin{cases} J_{p_i} & (i = 2) \\ 0 & (i \neq 2), \end{cases}$$

where  $J_{p_i}$  denotes the quotient ring  $\mathcal{O}_{S, p_i} / (f_i, \frac{\partial f_i}{\partial z_1}, \frac{\partial f_i}{\partial z_2})$  and  $(f_i, \frac{\partial f_i}{\partial z_1}, \frac{\partial f_i}{\partial z_2})$  is the ideal generated by  $f_i$  and the partial derivatives of  $f_i$ . Thus we have

$$\mathbb{H}^i(Q^\bullet) = \begin{cases} \oplus_i J_{p_i} & (i = 2) \\ 0 & (i \neq 2) \end{cases} \quad (9.5)$$

In particular,  $p_i$  is a node, then  $\mathcal{J}_{p_i} = \mathbb{C}$ . If an anti-canonical divisor  $D$  is a simple normal crossing divisor (nodes), then it is known that the hypercohomology groups  $\mathbb{H}^i(\Omega^\bullet(\log D))$  of the logarithmic complex is given by the singular cohomology groups  $H^i(S \setminus D)$  of the complement of the divisor. Thus we can calculate the Poisson cohomology groups by using the cohomology groups of the complement and each quotient ring  $J_{p_i}$ . In fact, the short exact sequence (9.4) yields the long exact sequence of the hypercohomology groups. Thus we have

$$\mathbb{H}^i(\hat{\Omega}^\bullet) \cong H^i(S \setminus D) \quad (i = 0, 1, 4).$$

If  $\mathbb{H}^3(\Omega^\bullet(\log D)) \cong H^3(S \setminus D) = \{0\}$ , then we have

$$0 \rightarrow H^2(S \setminus D) \rightarrow H^2(\hat{\Omega}^\bullet) \rightarrow \oplus_i J_{p_i} \rightarrow 0$$

**PROPOSITION 9.3.** *Let  $S$  be a complex surface with a Poisson structure  $\beta$ . We assume that the divisor  $D = \{\beta = 0\}$  has  $m$  nodes  $\{p_i\}_{i=1}^m$  and  $H^3(S \setminus D) = \{0\}$ . Then three cohomology groups  $\mathbb{H}^i(\hat{\Omega}^\bullet) \cong \mathbb{H}^i(\wedge^\bullet \Theta, \delta_\beta) \cong H^i(\wedge^\bullet \overline{L}_{\mathcal{J}_\beta})$  are given by*

$$\dim H^i(\wedge^\bullet \overline{L}_\beta) = \begin{cases} \dim H^2(S \setminus D) + m & (i = 2) \\ \dim H^i(S \setminus D) & (i \neq 2) \end{cases}$$

**REMARK 9.4.** If  $D$  is a cubic curve in  $\mathbb{CP}^2$  with only a node, then  $H^2(\wedge^\bullet \overline{L}_{\mathcal{J}_\beta}) = \mathbb{C}^2$ , however  $H^2(\mathbb{CP}^2 \setminus D) = \mathbb{C}$ . Thus the Lie algebroid cohomology groups of the singular  $D$  are different from the cohomology groups of the complement  $M \setminus D$ .

**EXAMPLE 9.5** (Del Pezzo surfaces). Let  $S_k = (M, J)$  be a del pezzo surface which is a blown up  $\mathbb{CP}^2$  at generic  $k$  points with ample anti-canonical line bundle, where  $0 \leq k \leq 8$ . If a Poisson structure  $\beta$  has a smooth divisor  $D$ , then we have  $\dim H^2(S_k \setminus D) = 2 + k$ . Thus we have

**PROPOSITION 9.6.** *If  $D$  is smooth, then  $\mathcal{J}_\beta$  has  $2 + k$  dimensional unobstructed deformations.*

The Poisson cohomology of  $S_k$  was already calculated in [HX]. Their calculation is different from ours. We shall follow their method. Let  $\beta$  be a holomorphic Poisson structure on  $S$  which is a section of  $K^{-1}$ . Since the Lie algebroid cohomology groups coincide with the hypercohomology groups of the Poisson complex :

$$0 \rightarrow \mathcal{O}_S \xrightarrow{\delta_\beta} \Theta_S \xrightarrow{\delta_\beta} \wedge^2 \Theta \rightarrow 0$$

The hypercohomology groups is the total cohomology groups of the double complex  $(\check{C}^p(\wedge^q \Theta), \delta_b, \delta)$ . Then the  $E_1$ -terms are given by the Čech cohomology groups :

$$\begin{array}{ccc} H^2(S, \mathcal{O}_S) & H^2(S, \Theta) & H^2(S, \wedge^2 \Theta) \\ H^1(S, \mathcal{O}_S) & H^1(S, \Theta) & H^1(S, \wedge^2 \Theta) \\ H^0(S, \mathcal{O}_S) & H^0(S, \Theta) & H^0(S, \wedge^2 \Theta) \end{array}$$

Applying the Kodaira vanishing theorem, we obtain that  $E_1$ -terms are given by

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & H^1(S, \Theta) & 0 \\ \mathbb{C} & H^0(S, \Theta) & H^0(S, \wedge^2 \Theta) \end{array}$$

The map  $\delta_\beta : H^0(S, \Theta) \rightarrow H^0(S, \wedge^2 \Theta)$  yields the  $E_2$ -terms, that is,

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & H^1(S, \Theta) & 0 \\ \mathbb{C} & \ker \delta_\beta & \text{image } \delta_\beta \end{array}$$

Thus it follows that the double complex degenerates at  $E_2$ -terms. A holomorphic vector field  $\alpha \in \ker \delta_\beta$  is given by  $\delta_\beta \alpha = -\mathcal{L}_\alpha \beta = 0$ . Thus  $\alpha \in \ker \delta_\beta$  gives an automorphism of  $S_k$  preserving a section  $\beta \in K^{-1}$ . The followings are known for  $S_k$

$$\begin{aligned} \dim H^0(S_k, \Theta) &= \begin{cases} 8 - 2k & (k = 0, 1, 2, 3) \\ 0 & (k = 4, 5, 6, 7, 8) \end{cases} \\ \dim H^1(S_k, \Theta) &= \begin{cases} 2k - 8 & (k = 5, 6, 7, 8) \\ 0 & (k \leq 4) \end{cases} \\ \dim H^0(S_k, K^{-1}) &= 10 - k \end{aligned}$$

Thus we obtain  $\mathbb{H}^i(\wedge^\bullet \Theta) \cong H^i(\wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}_\beta})$  by

$$\dim H^i(\wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}_\beta}) = \begin{cases} 1 & (i = 0) \\ \dim \ker \delta_\beta & (i = 1) \\ \dim H^1(S, \Theta) + \text{image } \delta_\beta & (k = 2) \\ 0 & (k \neq 0, 1, 2) \end{cases}$$

If no automorphism preserves an anti-canonical divisor  $D$ , then  $\dim \ker \delta_\beta = 0$ . In particular, if  $k = 4, 5, 6, 7, 8$ , then  $\dim H^i(\wedge^\bullet \overline{\mathcal{L}}_{\mathcal{J}_\beta})$  does not depend on a choice of  $\beta$ .

REMARK 9.7. The calculations as in Example 9.5 hold for degenerate del pezzo surfaces which are blown up  $\mathbb{CP}^2$  at the set of points in almost general position. Then  $K^{-1}$  is not ample, however we have  $H^1(K^{-1}) = \{0\}$  (c.f. [Go2]).

EXAMPLE 9.8 (Hirzebruch surfaces). Let  $F_e$  be the projective space bundle  $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-e))$  over  $\mathbb{CP}^1$  with  $e > 0$ , which is called Hirzebruch surface. Let  $f$  be a fibre of  $F_e$ . Then  $K^{-1}$  is given by  $2b + (e+2)f$ , where  $b$  is a section of  $F_e$  with  $b \cdot b = -e$ . Since  $K^{-1}$  is effective, we have the  $E_1$ -terms:

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & H^1(F_e, \Theta) & H^1(F_e, K^{-1}) \\ \mathbb{C} & H^0(F_e, \Theta) & H^0(F_e, K^{-1}) \end{array}$$

These are given by

$$\begin{aligned} \dim H^1(F_e, \Theta) &= e - 1, \quad \dim H^1(F_e, K^{-1}) = e - 3 \\ \dim H^0(F_e, \Theta) &= e + 5, \quad \dim H^0(F_e, K^{-1}) = e + 6, \end{aligned}$$

where  $\dim H^1(F_e, K^{-1}) = 0$  if  $e \leq 3$ . Let  $\beta$  be a poisson structure of  $F_e$  which gives a smooth anti-canonical divisor  $D$ . Then it turns out that

$$H^i(F_e \setminus D) = \begin{cases} \mathbb{C} & (i = 0) \\ \mathbb{C}^3 & (i = 2) \\ 0 & (i \neq 0, 2) \end{cases}$$

PROPOSITION 9.9. *Thus we obtain unobstructed deformations of  $\mathcal{J}_\beta$  which are parametrized by an open set of  $H^2(F_e \setminus D) \cong \mathbb{C}^3$ .*

Since the double complex degenerates at  $E_2$ -terms, it follows from Theorem 6.2 and  $H^1(F_e \setminus D) = H^3(F_e \setminus D) = \{0\}$  that the map between  $E_1$ -terms  $\delta_\beta^0 : H^0(F_e, \Theta) \rightarrow H^0(F_e, K^{-1})$  is injective and the map  $\delta_\beta^1 : H^1(F_e, \Theta) \rightarrow H^1(F_e, K^{-1})$  is surjective.

REMARK 9.10. Compared with unobstructed deformations of  $\mathcal{J}_\beta$ , it is remarkable that deformations of generalized complex structure starting from  $\mathcal{J}_J$  are always obstructed if  $e > 3$ . In fact, infinitesimal deformations of  $\mathcal{J}_J$  are  $H^1(F_e, \Theta) \oplus H^0(F_e, K^{-1})$  and the Kuranish map is given by the Schouten bracket, that is,  $\alpha + \beta \mapsto [\alpha + \beta, \alpha + \beta]_{\text{Sch}}$ , where  $\alpha \in H^1(F_e, \Theta)$  and  $\beta \in H^0(F_e, K^{-1})$ . Since  $H^2(F_e, \Theta) = \{0\}$ , the obstruction is  $2[\alpha, \beta]_{\text{Sch}} \in H^1(F_e, K^{-1})$  which is equal to  $2\delta_\beta^1(\alpha)$ . Since the map  $\delta_\beta^1 : H^1(F_e, \Theta) \rightarrow H^1(F_e, K^{-1})$  is surjective, there exists a  $\alpha \in H^1(F_e, \Theta)$  such that  $\delta_\beta(\alpha) \neq 0$ . Thus we have

PROPOSITION 9.11. *There always exists an obstruction to deformations of the generalized complex structure  $\mathcal{J}_J$  induced by the ordinary complex structures  $J$  on a Hirzebruch surface  $F_e$  if  $e > 3$ .*

### 9.3 $C^\infty$ logarithmic transformations and unobstructed deformations of generalized complex structures

The natural projection  $\wedge^\bullet T^*M \rightarrow \wedge^0 T^*M$  gives rise to a section  $s \in \Gamma(K_\mathcal{J}^*)$  on  $(M, \mathcal{J})$ . Then the zero set  $s^{-1}(0)$  is called *Type changing loci* of  $(M, \mathcal{J})$  which is the set of points of  $M$  where the Type number of  $\mathcal{J}$  changes from 0 to 2 if  $\mathcal{J}$  is a generalized complex structure of even type on a 4-manifold. We denote by  $\kappa(\mathcal{J})$  the number of connected components of Type changing loci of  $(M, \mathcal{J})$ . Let  $(M, \mathcal{J})$  be a generalized complex 4-manifold. A sub 2-torus  $T \subset M$  is a *symplectic torus* if there is a neighborhood  $\nu(T)$  such that the  $\mathcal{J}|_{\nu(T)}$  is induced from a real symplectic structure  $\omega$  and a  $b$ -field and  $T \subset (\nu(T), \omega)$  is a symplectic submanifold. Let  $T \subset (M, \mathcal{J})$  be a symplectic torus with trivial normal bundle. Then a neighborhood  $\nu(T)$  is diffeomorphic to  $D^2 \times T^2$  with the boundary  $\partial\nu(T) \cong T^3$ . A  $C^\infty$  logarithmic transformation on  $T$  consists of two operations: removing  $\nu(T)$  and attaching the manifold  $D^2 \times T^2$  by a diffeomorphism  $\Psi : \partial D^2 \times T^2 \rightarrow \partial\nu(T)$ . A  $C^\infty$  logarithmic transformation on  $T$  yields a manifold

$$M_\Psi := (M \setminus \text{Int}\nu(T)) \cup_\Psi (D^2 \times T^2)$$

where  $\text{Int}\nu(T)$  denotes the interior of  $\nu(T)$ . If  $\Psi(\partial D^2 \times pt) \subset D^2 \times T^2$  is null-homotopic in  $\nu(T)$ , the  $C^\infty$  logarithmic transformation is *trivial*. The boundary  $\partial\nu(T)$  is identified with  $\partial D^2 \times T^2$  and we denote by  $\pi$  the projection from  $\partial D^2 \times T^2$  to  $\partial D^2$ . Then we have the diagram

$$\begin{array}{ccc} \partial D^2 \times T^2 & \xrightarrow{\Psi} & \partial\nu(T^3) \\ & \searrow & \downarrow \pi \\ & & \partial D^2 \end{array}$$

and the map

$$\pi \circ \Psi|_{\partial D^2 \times \{pt\}} : \partial D^2 \times \{pt\} \rightarrow \partial D^2.$$

Then we define *multiplicity* of  $C^\infty$  logarithmic transformation to be the degree of the map  $\pi \circ \Psi|_{\partial D^2 \times \{pt\}} : S^1 \rightarrow S^1$

THEOREM 9.12. [GH] *Let  $(M, \mathcal{J})$  be a generalized complex 4-manifold and  $T$  a symplectic torus of  $(M, \mathcal{J})$  with trivial normal bundle. Then every nontrivial  $C^\infty$  logarithmic transformation yields a twisted generalized complex structure  $\mathcal{J}_\Psi$  on the manifold  $M_\Psi$  with  $\kappa(\mathcal{J}_\Psi) = \kappa(\mathcal{J}) + 1$ . In particular, if  $H^3(M_\Psi) = \{0\}$ , then  $\mathcal{J}_\Psi$  is a generalized complex structure.*

The proof of this theorem is already given in [GH]. For the completeness of the paper, we will give the proof of the theorem.

PROOF. If the logarithmic transformation determined by  $\Psi$  is trivial, the statement is obvious. We assume that the logarithmic transformation is not trivial.

Let  $\omega_T$  be a symplectic form of  $\nu T$  which induces the generalized complex structure  $\mathcal{J}|_{\nu T}$  and  $T$  is symplectic with respect to  $\omega_T$ . By Weinstein's neighborhood theorem [W], we can take a symplectomorphism:

$$\Theta : (\nu T, \omega_T) \rightarrow (D^2 \times T^2, \sigma_C),$$

where  $D^2$  denotes the unit disk  $\{z_1 \in \mathbb{C} \mid |z_1| \leq 1\}$  and  $T^2$  is the quotient  $\mathbb{C}/\mathbb{Z}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$  with a coordinate  $z_2$  and  $\sigma_C = \sqrt{-1}C(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$  for the constant  $C = \frac{1}{2} \int_T \omega_T$ . Using this identification, the attaching map  $\Psi$  can be regarded as a matrix  $A_\Psi \in \text{SL}(3; \mathbb{Z})$ . Any matrix  $P \in \text{SL}(2; \mathbb{Z})$  induces a self-diffeomorphism  $P : T^2 \rightarrow T^2$ . Since the map  $\text{id}_{D^2} \times P$  preserves the form  $\sigma_C$  and the diffeomorphism type of  $M_\Psi$  is determined by the first row of  $A_\Psi$ , we can assume that  $A_\Psi$  is equal to the following matrix:

$$\begin{pmatrix} m & 0 & p \\ 0 & 1 & 0 \\ a & 0 & b \end{pmatrix}$$

(see [GS]), where  $m, p, a, b$  are integers which satisfy  $mb - pa = 1$ . The first row of  $A_\Psi$  is  $(m, 0, p)$  and it follows that  $p \neq 0$  since  $\Psi$  is not trivial. Thus we can take  $a$  and  $b$  satisfying the condition

$$mbs - pa \neq 0 \quad \text{for all } s \in [0, 1] \quad (9.6)$$

by replacing  $(a, b)$  by  $(a + ml, b + pl)$  for a suitable integer  $l$  if necessary. Let  $D_k$  be the annulus  $\{z_1 \in \mathbb{C} \mid k < |z_1| \leq 1\}$  for  $k \in [0, 1]$ . We define a diffeomorphism  $\Psi : D_{\frac{1}{e}} \times T^2 \rightarrow D_0 \times T^2$  as follows:

$$\Psi(r, \theta_1, \theta_2, \theta_3) = (\sqrt{\log(er)}, m\theta_1 + a\theta_3, \theta_2, p\theta_1 + b\theta_3),$$

where  $z_1 = r \exp(\sqrt{-1}\theta_1)$  and  $z_2 = \theta_2 + \sqrt{-1}\theta_3$ . The manifold  $M_\Psi$  is diffeomorphic to the following manifold:

$$M_\Psi = (X \setminus \text{Int}(\nu T)) \cup_\Psi D^2 \times T^2.$$

Thus it suffices to construct a twisted generalized complex structure on  $M_\Psi$  which satisfies the conditions (9.1), (9.2) and (9.3)

We denote by  $\varphi_T := z_1 \phi_T \in \wedge^{\text{even}} D^2 \times T^2$  the following form:

$$z_1 \exp \left( -\frac{mC}{2} \varrho(|z_1|^2) \frac{dz_1}{z_1} \wedge \frac{d\bar{z}_1}{\bar{z}_1} - bC dz_2 \wedge d\bar{z}_2 + \frac{dz_1}{z_1} \wedge dw_2 \right), \quad (9.7)$$

where  $w_2 = (C(\frac{a}{2} - p)z_2 - C(\frac{a}{2} + p)\bar{z}_2)$  and  $\varrho : \mathbb{R} \rightarrow [0, 1]$  is a monotonic increasing function which satisfies  $\varrho(r) = 0$  if  $|r| < \frac{1}{2e^2}$  and  $\varrho(r) = 1$  if  $|r| \geq \frac{1}{e^2}$ . The form  $\varphi_T$  satisfies the condition (9.1). It follows from the condition (9.6) that the top-degree part  $\langle \varphi_T \wedge \overline{\varphi_T} \rangle$  is not trivial. Thus the condition (9.2) holds. Since  $z_1^{-1} \varphi_T$  is  $d$ -closed on  $D_0 \times T^2$ , the form  $\varphi_T$  satisfies the condition (9.3). Thus  $\varphi_T$  gives a generalized complex structure on  $D^2 \times T^2$ . Denote by  $B$  and  $\omega$  the real part and the imaginary part of the degree-2 part of the form  $\phi_T = z_1^{-1} \varphi_T$ , respectively. Then it follows from a direct calculation that the pullback  $\Psi^* \sigma_C$  is equal to  $\omega$ . We take a monotonic decreasing function  $\tilde{\varrho} : \mathbb{R} \rightarrow [0, 1]$  which satisfies  $\tilde{\varrho}(r) = 1$  for  $|r| < \frac{1}{2}$  and  $\tilde{\varrho}(r) = 0$  for  $|r| \geq 1 - \varepsilon$ , where  $\varepsilon > 0$  is a sufficiently small number. We define a 2-form  $\tilde{B} \in A^2(M \setminus T)$  by

$$\tilde{B} = \begin{cases} \Psi^{-1*}(\tilde{\varrho}(|z_1|^2)B) & \text{on } \nu T \setminus T, \\ 0 & \text{on } M \setminus \nu T. \end{cases}$$

Then the manifold  $M \setminus T$  admits a twisted generalized complex structure  $\mathcal{J}'_\Psi$  such that  $\exp(\tilde{B} + \sqrt{-1}\omega_T) \in \Omega^\bullet_C(\nu T \setminus T)$  is a local section of the canonical bundle  $K_{\mathcal{J}'_\Psi}$ . Since  $\varphi_T$  gives a generalized complex structure on  $D^2 \times T^2$ , the form  $\exp((\tilde{\varrho}(|z_1|^2) - 1)B)\varphi_T$  induces a  $(-d(\tilde{\varrho}(|z_1|^2)B))$ -twisted generalized complex structure. Since the 2-form  $\Psi^*(\tilde{B} + \sqrt{-1}\sigma_C)$  is equal to  $\tilde{\varrho}(|z_1|^2)B + \sqrt{-1}\omega$ , we obtain a twisted generalized complex structure on  $M_\Psi$  which satisfies the conditions (9.1), (9.2) and (9.3). If  $H^3(M_\Psi) = \{0\}$ , then there exists a 2-form  $\gamma \in A^2(M_\Psi)$  such that  $d\gamma = -d(\tilde{\varrho}(|z_1|^2)B)$ . Then  $\exp(\gamma + (\tilde{\varrho}(|z_1|^2) - 1)B)\varphi_T$  This completes the proof of Theorem 9.12.  $\square$

We can use logarithmic transformations of general multiplicity to obtain generalized complex structures with arbitrary large number of connected components of type changing loci.

**THEOREM 9.13.** [GH] *Let  $(M, \mathcal{J})$  be a generalized complex 4-manifold and  $T$  a torus with trivial normal bundle. We denote by  $M'$  the manifold obtained from  $M$  by a  $C^\infty$  logarithmic transformation on  $T$  of multiplicity 0. Then for every  $n > \kappa(\mathcal{J})$ ,  $M'$  admits a twisted generalized complex structure  $\mathcal{J}'_n$  with  $\kappa(\mathcal{J}'_n) = n$ . In particular, if  $H^3(M') = \{0\}$ , then  $\mathcal{J}_n$  is a generalized complex structure.*

**THEOREM 9.14.** [GH] *For every  $k, l \geq 0$  and  $m \geq 1$ , the connected sum  $(2k+1)\mathbb{C}P^2 \# l\overline{\mathbb{C}P}^2$  admits generalized complex structures  $\mathcal{J}_m$  with  $\kappa(\mathcal{J}_m) = m$ .*

**REMARK 9.15.** Cavalcanti and Gualtieri firstly constructed a generalized complex structure  $\mathcal{J}$  on the connected sum  $(2k+1)\mathbb{C}P^2 \# l\overline{\mathbb{C}P}^2$  with  $\kappa(\mathcal{J}) = 1$  by logarithmic transformations of multiplicity 0 which does not admit any complex structures and symplectic structures [CG1], [CG2].

**REMARK 9.16.** In [CG1] it is pointed out that another generalized complex surgery is possible, but little detail is provided. Working out the details of this another surgery one would obtain an alternative proof of Theorem 9.12 in the special cases where the first row of the matrix  $A_\Psi$  is  $(m, 0, 1)$ .

**REMARK 9.17.** Torres and Yazinski also constructed twisted generalized complex manifolds on several manifolds with arbitrary large type changing loci by a different method [TY].

We shall apply our theorems to these 4-manifolds obtained by  $C^\infty$  logarithmic transformations.

**THEOREM 9.18.** *Let  $(M, \mathcal{J})$  be a generalized complex structure and  $T$  a real symplectic torus with trivial normal bundle. We denote by  $(M_\Psi, \mathcal{J})$  a generalized complex 4-manifold constructed by a  $C^\infty$  logarithmic transformation from  $(M, \mathcal{J})$  along the torus  $T$  with type changing loci  $D$  as in Theorem 9.12, where we assume that  $H^3(M_\Psi) = \{0\}$ . Then the Lie algebroid cohomology  $H^k(\wedge^\bullet \overline{L}_{\mathcal{J}_\Psi})$  is isomorphic to  $H^k(M_\Psi \setminus D, \mathbb{C})$  and deformations of generalized complex structures of  $(M_\Psi, \mathcal{J}_\Psi)$  are unobstructed which are parametrized by an open set of  $H^2(M_\Psi \setminus D, \mathbb{C})$ .*

**PROOF.** It follows from (9.7) that the degree 2-part  $\omega_\mathbb{C}$  of  $\phi_T$  is given by

$$\left( -bC dz_2 \wedge d\bar{z}_2 + C \left( \frac{a}{2} - p \right) \frac{dz_1}{z_1} \wedge dz_2 - C \left( \frac{a}{2} + p \right) \frac{dz_1}{z_1} \wedge d\bar{z}_2 \right), \quad (9.8)$$

if  $r = |z_1| < \frac{1}{e^2}$ . On the complement of  $D$ , the  $\omega_\mathbb{C}$  is given by  $b + \sqrt{-1}\omega$  for a real 2-form  $b$  and a real symplectic form  $\omega$ . The 2-forms  $\omega_\mathbb{C}$  is written as

$$\frac{dz_1}{z_1} \wedge dw_2 - C' dw_2 \wedge d\bar{w}_2,$$

where  $w_2 = C(\frac{a}{2} - p)z_2 - C(\frac{a}{2} + p)\bar{z}_2$ , and  $C' = \frac{bc}{C^2(a^2 + 4p^2)}$ .

We define local  $C^\infty$  complex coordinates  $(w_1, w_2)$  by

$$w_1 = e^{C'\bar{w}_2} z_1, \quad w_2 = C(\frac{a}{2} - p)z_2 - C(\frac{a}{2} + p)\bar{z}_2.$$

Note that  $p$  is not equal to 0. Then we have

$$\omega_\mathbb{C} = \frac{dw_1}{w_1} \wedge dw_2.$$



Thus  $\omega_{\mathbb{C}}$  is a  $C^\infty$  logarithmic symplectic structure on a neighborhood of  $D$ .  $\mathcal{J}_\Psi$  is induced from  $\phi_T := e^{\kappa + \omega_{\mathbb{C}}}$ , where  $\kappa := \gamma + (\bar{\partial}(|z_1|^2) - 1)B$  is a  $d$ -closed real 2-form on  $M_\Psi$ . From Remark 3.1, the Lie algebroid cohomology groups are invariant under the action of  $d$ -closed  $b$ -fields. The action of  $d$ -closed  $b$ -fields also preserves the unobstructedness of deformations of generalized complex structure. Thus the results follow from Theorem 6.2 and Theorem 6.3.  $\square$

**THEOREM 9.19.** *Let  $\mathcal{J}_m$  be the generalized complex structure on the connected sum  $M := (2k - 1)\mathbb{CP}^2 \# (10k - 1)\overline{\mathbb{CP}^2}$  as in Theorem 9.14 and  $D$  the type changing loci of  $\mathcal{J}_m$ . Then the Lie algebroid cohomology  $H^k(\wedge^\bullet \overline{L}_{\mathcal{J}})$  is isomorphic to  $H^k(M \setminus D, \mathbb{C})$  and deformations of the generalized complex structure  $\mathcal{J}_m$  are unobstructed which are parametrized by an open set of  $H^2(M \setminus D, \mathbb{C})$ .*

**PROOF.** Let  $E(1)$  be the blown up  $\mathbb{CP}^2$  at 9 points of intersection of two generic cubic hyper-surfaces in  $\mathbb{CP}^2$ . The manifold  $E(1)$  is an elliptic fibration over  $\mathbb{CP}^1$ . We denote by  $E(k)$  the fibre sum of  $k$  copies of  $E(1)$ . Then a  $C^\infty$  logarithmic transformation of multiplicity 0 along a smooth fibre of  $E(k)$  gives a manifold  $M := (2k - 1)\mathbb{CP}^2 \# (10k - 1)\overline{\mathbb{CP}^2}$ . As in [GH], we can apply a  $C^\infty$  logarithmic transformations of multiplicity 1 on fibres of fishtail neighborhood at  $m$  times so that the diffeomorphism type of  $M := (2k - 1)\mathbb{CP}^2 \# (10k - 1)\overline{\mathbb{CP}^2}$  is not changed. The procedure gives the generalized complex structure  $\mathcal{J}_m$  on  $M := (2k - 1)\mathbb{CP}^2 \# (10k - 1)\overline{\mathbb{CP}^2}$  such that  $\kappa(\mathcal{J}_m) = m$ . Thus it follows from Theorem 9.18 that  $\mathcal{J}_m$  is induced from a logarithmic symplectic structure acted by a  $d$ -closed  $b$ -field. The result follows from Theorem 6.2 and Theorem 6.3.  $\square$

**REMARK 9.20.** Let  $D = \cup_{i=1}^m T_i$  be the type changing loci of generalized complex structure  $\mathcal{J}_m$  in Theorem 9.19. Then  $E(n) \setminus D \cong M \setminus D$  and we obtain an exact sequence

$$H_D^2((E(k)) \xrightarrow{i} H^2(E(n)) \rightarrow H^2(E(n) \setminus D) \rightarrow H_D^3(E(n)) \rightarrow 0,$$

where  $H_D^i(E(k))$  denotes the local cohomology of  $E(k)$  with support  $D$ . By the duality, we have  $H_D^i(E(k)) \cong H^{i-2}(D)$ . The image of the map  $i : H_D^2((E(k)) \rightarrow H^2(E(n))$  is a one dimensional subspace of  $H^2(E(k))$  which is generated by the 1-st Chern class of the line bundle given by a fibre of  $E(n)$ . Thus we have  $\dim H^2(M \setminus D) = \dim H^2(E(k)) + \dim H^1(D) - 1$ . Since  $\dim H^2(E(k)) = 12k - 2$ , it follows that  $\dim H^2(M \setminus D) = 12k + 2m - 3$ .

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